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Quantum (2+1) kinematical algebras: a global approach

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Abstract. In this paper we give an approach to quantum deformations of the (2+1) kinematical Lie algebras within a scheme that simultaneously describes all groups of motions of classical geometries in $N = 3$ dimensions. We cover at once all the kinematical geometries including the quantum versions of Inönü–Wigner contractions, which are defined in a natural way and relate q-deformations as expected. We thus obtain some q-deformations previously known for the three-dimensional Euclidean and (2+1)–Poincaré algebras and also some new q-deformations for these and other kinematical algebras, such as the (2+1)–de Sitter, Galilei and Newton–Hooke algebras.

1. Introduction

The deformation of the universal enveloping algebras of classical simple Lie algebras can be now considered as a solved problem in quantum group theory [1–3]. However, many physically interesting algebras (for instance, Galilei and Poincaré algebras) do not belong to this class. They are neither orthogonal nor pseudo-orthogonal real forms of classical simple Lie algebras, but are related to them by means of Inönü–Wigner (IW) contractions [4]. Among the methods for obtaining quantum analogues of kinematical algebras [5–7], a successful one is based on a generalization of such a contraction procedure [8].

The task of giving a comprehensive framework summarizing all possible classes of 'quantum contractions' is not a straightforward consequence of the IW concept of contraction. Any transformation acting on a quantum group has to be defined on two different (but deeply related) levels: the algebra and co-algebra structures (plus the antipode) [9, 10]. Classical contractions deal with the former level, but their quantum generalizations, which concern both levels together, are no longer unique and the number of different possibilities increases strongly with the algebra dimension.

The present paper uses a Cayley–Klein (CK) type geometrical setting [11–15] as a supplementary structure in order to make some progress with this problem. Although the geometrical interpretation is lost when quantizing, the underlying classical structure remains relevant in terms of properties of the Hopf algebra.

In section 2 we give a brief overview about those classical features of the CK Lie algebras appearing in classical three-dimensional CK geometries (3D-CKG) that keep their interest in the quantum case. Section 3 presents a prescription for a simultaneous q-deformation of all these CK algebras. A matrix realization is also given. In section 4 some quantum (2+1) kinematical algebras appearing in our scheme are studied: three (2+1) q-Poincaré and q-de Sitter algebras, a (2+1) q-Galilei algebra and a (2+1) q-Newton–Hooke algebra. Most of these q-algebras are new, but some are already known (the (2+1) q-Poincaré given by Lukierski *et al* in [16] and the 3D q-Euclidean obtained by Celeghini *et al* [17]). Finally, in the last section we make some remarks and comments.

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2. The three-dimensional Cayley–Klein geometries

The goal in this section is not so much to give a complete geometrical description of the 3D-CKG as to introduce a global picture and to collect those properties of their motion groups that will be used later. A complete description of these 'classical' systems for arbitrary dimension will be given in a forthcoming paper [13]. From a group theoretical viewpoint, a 3D-CKG is determined by a six-dimensional Lie group G and a set of three commuting involutions $S_{(0)}$, $S_{(1)}$ and $S_{(2)}$ of the Lie algebra $\mathfrak{g}$ of G such that:

(1) The invariant elements of $\mathfrak{g}$ under each involution are subalgebras $\mathfrak{h}^{(0)}$, $\mathfrak{h}^{(1)}$ and $\mathfrak{h}^{(2)}$ of $\mathfrak{g}$ of dimensions 3, 2 and 3, respectively.
(2) The group G acts transitively on the symmetric homogeneous spaces $\mathcal{X}^{(i)} \equiv G/H^{(i)}$ ($i = 0, 1, 2$), where $H^{(i)}$ is the Lie subgroup of G with Lie subalgebra $\mathfrak{h}^{(i)}$. The homogeneous spaces $\mathcal{X}^{(i)}$ ($i = 0, 1, 2$) are identified with the spaces of (first kind) points, lines and planes, respectively.
(3) The 2D-subgeometries associated to the sets of lines through a point of $\mathcal{X}^{(0)}$ and to the set of planes through a point of $\mathcal{X}^{(0)}$ are 2D-CKG.

These conditions completely characterize all the 3D-CKG via their motion groups. Let J_{ij}, ($i < j$; $i, j = 0, 1, 2, 3$) be a basis of the Lie algebra $\mathfrak{g}$, such that the action of the involutions $S_{(i)}$ ($i = 0, 1, 2$) is defined as

$$
\begin{aligned}
S_{(0)} &: (J_{01}, J_{02}, J_{03}, J_{12}, J_{13}, J_{23}) \longrightarrow (-J_{01}, -J_{02}, -J_{03}, J_{12}, J_{13}, J_{23}) \\
S_{(1)} &: (J_{01}, J_{02}, J_{03}, J_{12}, J_{13}, J_{23}) \longrightarrow (J_{01}, -J_{02}, -J_{03}, -J_{12}, -J_{13}, J_{23}) \\
S_{(2)} &: (J_{01}, J_{02}, J_{03}, J_{12}, J_{13}, J_{23}) \longrightarrow (J_{01}, J_{02}, -J_{03}, J_{12}, -J_{13}, -J_{23}) .
\end{aligned}
\tag{2.1}
$$

These involutions generate a $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$ Abelian group and originate a grading of the Lie algebra $\mathfrak{g}$ [18–21]. Hypothesis (1) is then automatically fulfilled; by now imposing hypotheses (2) and (3), the Lie algebra $\mathfrak{g}$ turns out to be determined by three real parameters κ_i ($i = 1, 2, 3$); it will be denoted $\mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)}$ and its Lie brackets can be written in terms of the parameters $\kappa_{ij} := \kappa_{i+1}\kappa_{i+2}\dots\kappa_j$ ($i < j$; $i, j = 0, 1, 2, 3$) as follows

$$
[J_{ij}, J_{lm}] = \delta_{im} J_{lj} - \delta_{jl} J_{im} + \delta_{jm}\kappa_{lm} J_{il} + \delta_{il}\kappa_{ij} J_{jm} \qquad i \leqslant l \quad j \leqslant m . \tag{2.2}
$$

There are two linearly independent second-order Casimirs for $\mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)}$:

$$
\begin{aligned}
\mathcal{C}_1 &= \kappa_2\kappa_3 J_{01}^2 + \kappa_3 J_{02}^2 + J_{03}^2 + \kappa_1\kappa_3 J_{12}^2 + \kappa_1 J_{13}^2 + \kappa_1\kappa_2 J_{23}^2 \\
\mathcal{C}_2 &= J_{03} J_{12} + \kappa_2 J_{01} J_{23} - J_{02} J_{13} .
\end{aligned}
\tag{2.3}
$$

The parameters κ_i ($i = 1, 2, 3$) are linked to the 'kind' of measure of separation between points, lines and planes (elliptical, parabolical or hyperbolical in Klein terminology according to κ_i is > 0, $= 0$ or < 0, respectively). Each κ_i can be separately scaled to $+1$, 0, or -1, so there are $3^3 = 27$ essentially different CKG associated to the possible triads $(\kappa_1, \kappa_2, \kappa_3)$ with $\kappa_i \in \{+, 0, -\}$. The same abstract Lie algebra may appear more than once in the list $\mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)}$. When *all* κ_i are different from zero (eight cases), the algebras (2.2) are $so(4)$ (only once for (+,+,+)), $so(3,1)$ (four times) and $so(2,2)$ (three times). When one or more κ_i are equal to zero, we obtain the so-called quasisimple algebras; among them we get the inhomogeneous orthogonal algebras $iso(3)$ Euclidean algebra (twice), $iso(2,1)$ Poincaré algebra (six times) and $iiso(2)$ Galilean algebra (twice). This multiple appearance

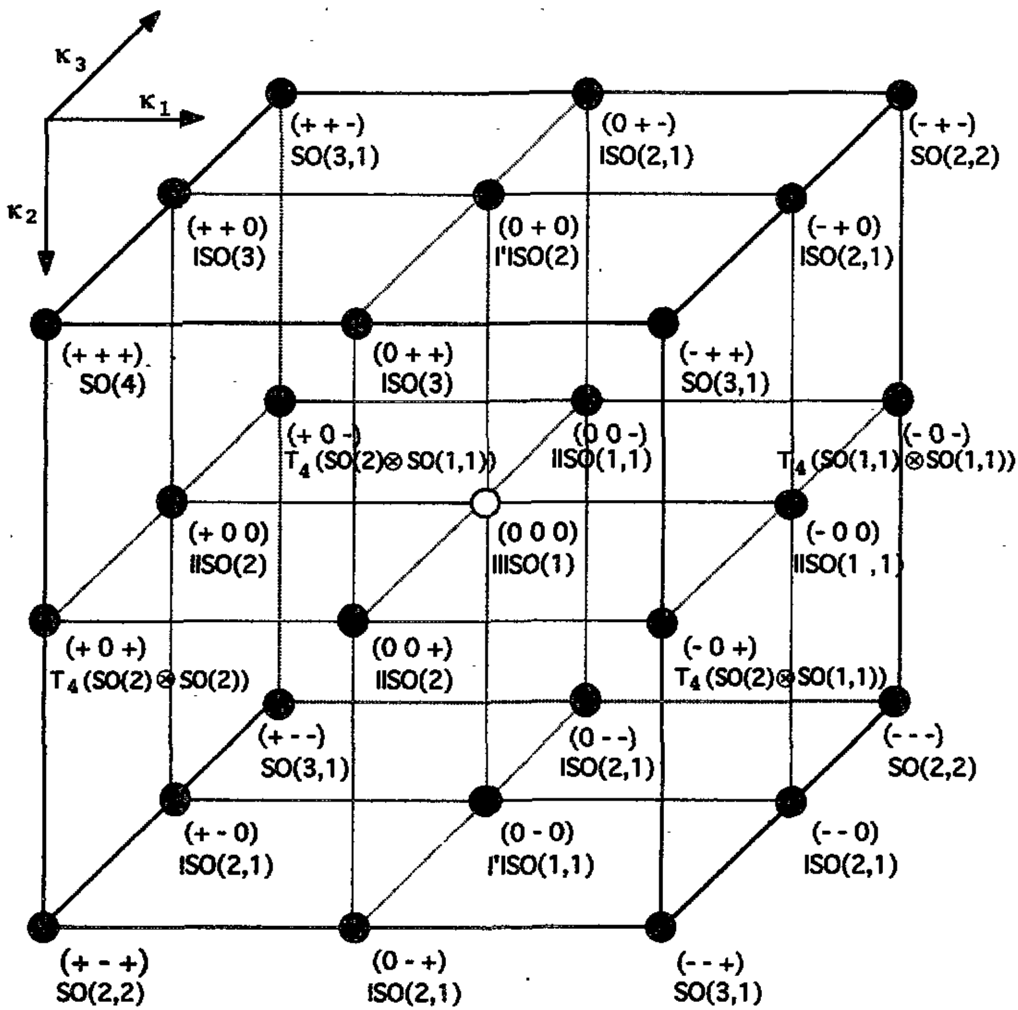

Figure 1. Groups of motion of the 3D-CKG. Each point corresponds to a set of values $(\kappa_1, \kappa_2, \kappa_3)$ of the fundamental constants. The group G with Lie algebra $\mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)}$ is also shown. Contractions around points, (resp. lines and planes) move from the sides to the centre slice of the cube, following the direction κ_1 (resp. κ_2, κ_3).

has a clear geometrical interpretation [13]. Figure 1 displays these 3D-CKG and gives for each value of $(\kappa_1, \kappa_2, \kappa_3)$ their Lie groups as abstract groups.

Each (basic) involution $S_{(i)}$ gives rise to an IW contraction by leaving fixed the elements invariant under such involution, multiplying the anti-invariant elements by a parameter ε, and taking the limit $\varepsilon \to 0$. We have three basic contractions:

$$\begin{aligned}
S_{(0)} &\leftrightarrow \text{Local contr } (\mathbb{J}_{01}, \mathbb{J}_{02}, \mathbb{J}_{03}, \mathbb{J}_{12}, \mathbb{J}_{13}, \mathbb{J}_{23}) \equiv (\varepsilon J_{01}, \varepsilon J_{02}, \varepsilon J_{03}, J_{12}, J_{13}, J_{23}) \\
S_{(1)} &\leftrightarrow \text{Axial contr } (\mathbb{J}_{01}, \mathbb{J}_{02}, \mathbb{J}_{03}, \mathbb{J}_{12}, \mathbb{J}_{13}, \mathbb{J}_{23}) \equiv (J_{01}, \varepsilon J_{02}, \varepsilon J_{03}, \varepsilon J_{12}, \varepsilon J_{13}, J_{23}) \qquad (2.4) \\
S_{(2)} &\leftrightarrow \text{Planar contr } (\mathbb{J}_{01}, \mathbb{J}_{02}, \mathbb{J}_{03}, \mathbb{J}_{12}, \mathbb{J}_{13}, \mathbb{J}_{23}) \equiv (J_{01}, J_{02}, \varepsilon J_{03}, J_{12}, \varepsilon J_{13}, \varepsilon J_{23})
\end{aligned}$$

where $\mathbb{J}_{ij}$ denotes the new generators which close the contracted algebra. The contraction linked with the involution $S_{(i)}$ $(i = 0, 1, 2)$ makes zero the parameter κ_{i+1} while keeping invariant the others and originates a new (contracted) geometry that describes the behaviour of the old one in a neighbourhood of a point, a line or a plane, respectively.

The 3D-CK algebras can be realized in terms of 4×4 real matrices,

$$\mathcal{D}(J_{ij}) = -\kappa_{ij} e_{ij} + e_{ji} \tag{2.5}$$

where e_{ij} are 4×4 matrices with elements $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and commutation relations

$$[e_{ij}, e_{lm}] = \delta_{jl} e_{im} - \delta_{im} e_{lj} . \tag{2.6}$$

This matrix realization $\mathcal{D}$ allows us to consider the CK groups as groups of linear transformations on $\mathbb{R}^4$. The one-parameter subgroup associated to the generator J_{ij} can be written explicitly using the 'generalized' trigonometric functions $\mathrm{sine}_\kappa(x) \equiv S_\kappa(x)$ and $\mathrm{cosine}_\kappa(x) \equiv C_\kappa(x)$ [12, 14], defined in terms of power series as follows:

$$S_\kappa(x) = \sum_{l=0}^{\infty} (-\kappa)^l \frac{x^{2l+1}}{(2l+1)!} \qquad C_\kappa(x) = \sum_{l=0}^{\infty} (-\kappa)^l \frac{x^{2l}}{(2l)!} . \tag{2.7}$$

The standard circular and hyperbolic functions are recovered when $\kappa = 1$ and $\kappa = -1$, respectively. The case $\kappa = 0$ corresponds to the 'parabolic' or galilean trigonometric functions $(S_0(x) = x,\ C_0(x) = 1)$. Some particular properties of these functions that will be used in the next section are

$$\begin{gathered} C_\kappa^2(x) + \kappa S_\kappa^2(x) = 1 \qquad C_\kappa(x \pm y) = C_\kappa(x) C_\kappa(y) \mp \kappa S_\kappa(x) S_\kappa(y) \\ S_\kappa(x \pm y) = S_\kappa(x) C_\kappa(y) \pm C_\kappa(x) S_\kappa(y) \end{gathered} \tag{2.8}$$

and also

$$C_\kappa(x) = \frac{\mathrm{e}^{\mathrm{i}\sqrt{\kappa}x} + \mathrm{e}^{-\mathrm{i}\sqrt{\kappa}x}}{2} \qquad S_\kappa(x) = \frac{\mathrm{e}^{\mathrm{i}\sqrt{\kappa}x} - \mathrm{e}^{-\mathrm{i}\sqrt{\kappa}x}}{2i\sqrt{\kappa}} . \tag{2.9}$$

Another interesting feature of CKG is duality. This is a geometric transformation which carries points, lines and planes of a 3D-CKG into planes, lines and points, respectively, of a 3D-CKG called the *dual* of the original one. In terms of the Lie algebra $\mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)}$, duality is an automorphism $(D_0 : J_{ij} \to \mathbb{J}_{ij})$ of $\mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)}$ given by:

$$(\mathbb{J}_{01}, \mathbb{J}_{02}, \mathbb{J}_{03}, \mathbb{J}_{12}, \mathbb{J}_{13}, \mathbb{J}_{23}) = (-J_{23}, -J_{13}, -J_{03}, -J_{12}, -J_{02}, -J_{01}) . \tag{2.10}$$

The 3D-CKG generated by the new Lie algebra $D_0 \mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)}$ is determined by a new set of parameters κ_i'. It is easy to check that

$$D_0 : \mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)} \to \mathfrak{g}_{(\kappa_3,\kappa_2,\kappa_1)} . \tag{2.11}$$

The action of D_0 over the whole system of 3D-CKG is displayed in figure 2. The geometries represented by full circles are interchanged according to the arrows while the open circles are auto-dual.

Finally, we mention that the concept of duality can be extended: in three dimensions there are 24 dualities, which include D_0 as a particular case [11, 13].

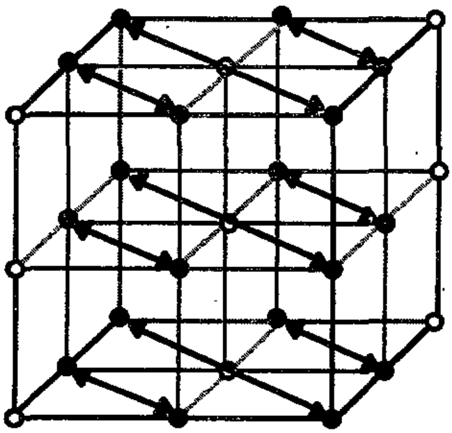

Figure 2. Ordinary duality in CKG. It can be described as the reflection in the diagonal plane $\kappa_1 = \kappa_3$.

3. The quantum Cayley–Klein algebras

Let $\mathfrak{g}$ be a Lie algebra. Let us consider a completion A of its universal enveloping algebra $U\mathfrak{g}$ built as formal power series in a deformation parameter z with coefficients in $U\mathfrak{g}$. We obtain a quantum deformation (in Drinfeld's sense [2]) of $U\mathfrak{g}$ by endowing this associative completion with a (deformed) Hopf structure [9]. The Hopf algebra structure very sharply restricts the possibilities for the formal power series [22]. We have to define the co-product ($\Delta : A \longrightarrow A \otimes A$) and co-unit ($\epsilon : A \longrightarrow \mathbb{C}$), as well as the antipode ($\gamma : A \longrightarrow A$) such that, $\forall a \in A$:

$$(id \otimes \Delta)\Delta(a) = (\Delta \otimes id)\Delta(a) \tag{3.1}$$

$$(id \otimes \epsilon)\Delta(a) = (\epsilon \otimes id)\Delta(a) = a \tag{3.2}$$

$$m((id \otimes \gamma)\Delta(a)) = m((\gamma \otimes id)\Delta(a)) = \epsilon(a)1 \tag{3.3}$$

where m is the usual multiplication $m(a \otimes b) = ab$. The q-algebra $U_q\mathfrak{g}$ ($q = e^z, z \in \mathbb{C}$) is completely defined once the deformed commutation relations are also given. Since the co-product Δ is an algebra homomorphism these relations have to be consistent with it, and the 'classical' Lie structure (as Hopf algebra) has to be recovered in the limit $z \to 0$.

The CK approach to quantum three-dimensional algebras [14] (see [23, 24] for a similar scheme) contains a leading idea: to look for a simultaneous quantization of all the CK algebras in such a way that, whatever any particular value of the κ_i parameters, there is always a non-trivial deformed algebra structure. Our point of view presents two main advantages: the global perspective clarifies the role of the different possible contractions and the pattern of quantization assures that a deformed Casimir element is always available (compare [23]); the appealing interpretation of the quantum inhomogeneous algebras as symmetries of discrete systems [25–28] is based on the existence of such a Casimir leading to a differential–difference kinematical equation.

3.1. The Hopf algebra $U_q\mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)}$

In the following, we give a quantization for the (2+1)-dimensional case fulfilling the above requirements. This fact is strongly related with the selection of J_{03} and J_{12} as a pair of

commuting primitive generators. The quantized universal enveloping CK algebra $U_q\mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)}$ is defined by the following.
(i) The deformed co-product:

$$
\begin{aligned}
\Delta(J_{03}) &= 1\otimes J_{03} + J_{03}\otimes 1 \qquad \Delta(J_{12}) = 1\otimes J_{12} + J_{12}\otimes 1\\
\Delta(J_{01}) &= \mathrm{e}^{-\frac{1}{2}zJ_{03}} C_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\otimes J_{01} + J_{01}\otimes \mathrm{e}^{\frac{1}{2}zJ_{03}} C_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\\
&\quad - \mathrm{e}^{-\frac{1}{2}zJ_{03}} S_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\otimes \kappa_1 J_{23} + \kappa_1 J_{23}\otimes \mathrm{e}^{\frac{1}{2}zJ_{03}} S_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\\
\Delta(J_{02}) &= \mathrm{e}^{-\frac{1}{2}zJ_{03}} C_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\otimes J_{02} + J_{02}\otimes \mathrm{e}^{\frac{1}{2}zJ_{03}} C_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\\
&\quad + \mathrm{e}^{-\frac{1}{2}zJ_{03}} S_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\otimes \kappa_1 J_{13} - \kappa_1 J_{13}\otimes \mathrm{e}^{\frac{1}{2}zJ_{03}} S_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\\
\Delta(J_{13}) &= \mathrm{e}^{-\frac{1}{2}zJ_{03}} C_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\otimes J_{13} + J_{13}\otimes \mathrm{e}^{\frac{1}{2}zJ_{03}} C_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\\
&\quad + \mathrm{e}^{-\frac{1}{2}zJ_{03}} S_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\otimes \kappa_3 J_{02} - \kappa_3 J_{02}\otimes \mathrm{e}^{\frac{1}{2}zJ_{03}} S_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\\
\Delta(J_{23}) &= \mathrm{e}^{-\frac{1}{2}zJ_{03}} C_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\otimes J_{23} + J_{23}\otimes \mathrm{e}^{\frac{1}{2}zJ_{03}} C_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\\
&\quad - \mathrm{e}^{-\frac{1}{2}zJ_{03}} S_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\otimes \kappa_3 J_{01} + \kappa_3 J_{01}\otimes \mathrm{e}^{\frac{1}{2}zJ_{03}} S_{-\kappa_1\kappa_3}(\tfrac{1}{2}zJ_{12})\,.
\end{aligned}
\tag{3.4}
$$

(ii) The co-unit

$$\epsilon(J_{ij}) = 0 \qquad i,j = 0,1,2,3, \quad i<j\,. \tag{3.5}$$

(iii) The antipode

$$\gamma(J_{ij}) = -\mathrm{e}^{zJ_{03}} J_{ij} e^{-zJ_{03}} \qquad i,j=0,1,2,3, \quad i<j \tag{3.6}$$

which can be explicitly computed and written in matrix form as

$$
\gamma\begin{pmatrix} J_{01}\\ J_{02}\\ J_{03}\\ J_{12}\\ J_{13}\\ J_{23}\end{pmatrix} = -\begin{pmatrix}
C_{\kappa_{03}}(z) & 0 & 0 & 0 & -\kappa_1 S_{\kappa_{03}}(z) & 0\\
0 & C_{\kappa_{03}}(z) & 0 & 0 & 0 & -\kappa_1\kappa_2 S_{\kappa_{03}}(z)\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
\kappa_2\kappa_3 S_{\kappa_{03}}(z) & 0 & 0 & 0 & C_{\kappa_{03}}(z) & 0\\
0 & \kappa_3 S_{\kappa_{03}}(z) & 0 & 0 & 0 & C_{\kappa_{03}}(z)
\end{pmatrix}\begin{pmatrix} J_{01}\\ J_{02}\\ J_{03}\\ J_{12}\\ J_{13}\\ J_{23}\end{pmatrix}
\tag{3.7}
$$

where we recall that $\kappa_{03} = \kappa_1\kappa_2\kappa_3$.
(iv) The non-vanishing commutation relations are

$$
\begin{aligned}
&[J_{12}, J_{01}] = J_{02} \qquad [J_{12}, J_{02}] = -\kappa_2 J_{01} \qquad [J_{01}, J_{02}] = \kappa_1 S_{-z^2\kappa_1\kappa_3}(J_{12}) C_{-z^2}(J_{03})\\
&[J_{13}, J_{01}] = S_{-z^2}(J_{03}) C_{-z^2\kappa_1\kappa_3}(J_{12}) \qquad [J_{13}, J_{03}] = -\kappa_2\kappa_3 J_{01} \qquad [J_{01}, J_{03}] = \kappa_1 J_{13}\\
&[J_{23}, J_{02}] = S_{-z^2}(J_{03}) C_{-z^2\kappa_1\kappa_3}(J_{12}) \qquad [J_{23}, J_{03}] = -\kappa_3 J_{02} \qquad [J_{02}, J_{03}] = \kappa_1\kappa_2 J_{23}\\
&[J_{23}, J_{12}] = J_{13} \qquad [J_{23}, J_{13}] = -\kappa_3 S_{-z^2\kappa_1\kappa_3}(J_{12}) C_{-z^2}(J_{03}) \qquad [J_{12}, J_{13}] = \kappa_2 J_{23}\,.
\end{aligned}
\tag{3.8}
$$

Notice that (2.9) assures that, for $C_\kappa(x)$, $S_\kappa(x) \in A$, $\Delta C_\kappa(x)$, $\Delta S_\kappa(x) \in A\otimes A$ (see [22]). For instance

$$
\begin{aligned}
\Delta S_\kappa(x) &= S_\kappa(x)\otimes C_\kappa(x) + C_\kappa(x)\otimes S_\kappa(x)\\
\Delta C_\kappa(x) &= C_\kappa(x)\otimes C_\kappa(x) - \kappa S_\kappa(x)\otimes S_\kappa(x)
\end{aligned}
\tag{3.9}
$$

(compare with (2.8)).

The quantum analogues of the second-order Casimirs (2.3) have the following form:

$$
\begin{aligned}
C_1^q &= 4C_{\kappa_{03}}(z)\left[S^2_{-z^2}(\tfrac{1}{2}J_{03})C^2_{-z^2\kappa_1\kappa_3}(\tfrac{1}{2}J_{12}) + \kappa_1\kappa_3 S^2_{-z^2\kappa_1\kappa_3}(\tfrac{1}{2}J_{12})C^2_{-z^2}(\tfrac{1}{2}J_{03})\right] \\
&\qquad + (1/z)S_{\kappa_{03}}(z)\left[\kappa_2\kappa_3 J_{01}^2 + \kappa_3 J_{02}^2 + \kappa_1 J_{13}^2 + \kappa_1\kappa_2 J_{23}^2\right] \\
C_2^q &= C_{\kappa_{03}}(z)S_{-z^2}(J_{03})S_{-z^2\kappa_1\kappa_3}(J_{12}) + (1/z)S_{\kappa_{03}}(z)[\kappa_2 J_{01}J_{23} - J_{02}J_{13}]\,.
\end{aligned}
\tag{3.10}
$$

When all the κ_i are different from zero, no Hopf subalgebras exist except the trivial (Lie) one generated by $\langle J_{03}, J_{12}\rangle$, but for quasi-simple q-algebras some Hopf subalgebras arise and will be separately studied for each specific case. Of course, the limit $z \to 0$ of (3.4)–(3.8), (3.10) leads to the classical Hopf algebra $U\mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)}$ and to the classical Casimirs.

The matrix realization $\mathcal{D}(J_{ij})$ of the classical CK algebras given by (2.5) can be implemented to a matrix realization $\mathcal{D}_q(J_{ij})$ of the q-CK algebras with J_{03} and J_{12} represented again by the matrices (2.5): after a straightforward computation we obtain

$$
\begin{aligned}
&S_{-z^2}(\mathcal{D}(J_{03})) = (1/z)S_{\kappa_{03}}(z)\,\mathcal{D}(J_{03}) \\
&C_{-z^2}(\mathcal{D}(J_{03})) = I + V_{\kappa_{03}}(z)\,\mathcal{D}^2(J_{03}) \\
&S_{-z^2\kappa_1\kappa_3}(\mathcal{D}(J_{12})) = (1/z)S_{\kappa_{03}}(z)\,\mathcal{D}(J_{12}) \\
&C_{-z^2\kappa_1\kappa_3}(\mathcal{D}(J_{12})) = I + \kappa_1\kappa_3 V_{\kappa_{03}}(z)\,\mathcal{D}^2(J_{12})
\end{aligned}
\tag{3.11}
$$

where I is the identity matrix and $V_\kappa(z) = (1 - C_\kappa(z))/\kappa$ is the general version of the old 'versus sinus' [12]. We get the quantum matrix realization as

$$
\begin{aligned}
&\mathcal{D}_q(J_{ij}) = \sqrt{(1/z)S_{\kappa_{03}}(z)}\,\mathcal{D}(J_{ij}) \qquad \text{if} \quad ij = 01, 02, 13, 23 \\
&\mathcal{D}_q(J_{ij}) = \mathcal{D}(J_{ij}) \qquad \text{if} \quad ij = 03, 12\,.
\end{aligned}
\tag{3.12}
$$

Notice that when *any* of the parameters κ_i is equal to zero, this representation coincides with the classical one.

3.2. Quantum involutions, contractions and dualities

A large part of the classical structure of involutions, contractions and dualities underlying CKG can be generalized for the quantum case if we allow the deformation parameter z (which becomes a dimensional quantity) to be transformed. This aspect and its relation with a possible physical meaning of z, given in [25–28], has been studied in detail in the (1+1)-dimensional case [14].

The three basic involutions in $U_q\mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)}$ are defined by

$$
\begin{aligned}
S^q_{(0)}&: (J_{01}, J_{02}, J_{03}, J_{12}, J_{13}, J_{23}; z) \longrightarrow (-J_{01}, -J_{02}, -J_{03}, J_{12}, J_{13}, J_{23}; -z) \\
S^q_{(1)}&: (J_{01}, J_{02}, J_{03}, J_{12}, J_{13}, J_{23}; z) \longrightarrow (J_{01}, -J_{02}, -J_{03}, -J_{12}, -J_{13}, J_{23}; -z) \\
S^q_{(2)}&: (J_{01}, J_{02}, J_{03}, J_{12}, J_{13}, J_{23}; z) \longrightarrow (J_{01}, J_{02}, -J_{03}, J_{12}, -J_{13}, -J_{23}; -z)
\end{aligned}
\tag{3.13}
$$

and generate an Abelian group $(\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2)$ leaving invariant the entire Hopf structure.

Three basic contractions are linked with the involutions $S^q_{(i)}$ and are defined in terms of IW contractions by the following transformations depending on a new parameter ε:

$$\begin{aligned}
&S^q_{(0)} \leftrightarrow q\text{-Local contr } (\mathbb{J}_{01}, \mathbb{J}_{02}, \mathbb{J}_{03}, \mathbb{J}_{12}, \mathbb{J}_{13}, \mathbb{J}_{23}; w) \equiv (\varepsilon J_{01}, \varepsilon J_{02}, \varepsilon J_{03}, J_{12}, J_{13}, J_{23}; z/\varepsilon) \\
&S^q_{(1)} \leftrightarrow q\text{-Axial contr } (\mathbb{J}_{01}, \mathbb{J}_{02}, \mathbb{J}_{03}, \mathbb{J}_{12}, \mathbb{J}_{13}, \mathbb{J}_{23}; w) \equiv (J_{01}, \varepsilon J_{02}, \varepsilon J_{03}, \varepsilon J_{12}, \varepsilon J_{13}, J_{23}; z/\varepsilon) \\
&S^q_{(2)} \leftrightarrow q\text{-Plane contr } (\mathbb{J}_{01}, \mathbb{J}_{02}, \mathbb{J}_{03}, \mathbb{J}_{12}, \mathbb{J}_{13}, \mathbb{J}_{23}; w) \equiv (J_{01}, J_{02}, \varepsilon J_{03}, J_{12}, \varepsilon J_{13}, \varepsilon J_{23}; z/\varepsilon)
\end{aligned} \tag{3.14}$$

where $\mathbb{J}_{ij}$ and w are the transformed generators and the new deformation parameter. Applying the transformation (3.14) associated to $S^q_{(i)}$ to (3.4)–(3.8) and making the limit $\varepsilon \to 0$ we get a q-CK algebra with $\kappa_{i+1} = 0$ and the remaining κ_i unchanged. An explicit example will be computed in section 4.

Duality can be implemented in $U_q\mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)}$ by preserving its 'classical' action over the generators J_{ij} and coefficients κ_i and adding up a transformation law for z:

$$D^q_0 : (\mathbb{J}_{01}, \mathbb{J}_{02}, \mathbb{J}_{03}, \mathbb{J}_{12}, \mathbb{J}_{13}, \mathbb{J}_{23}; w) \equiv (-J_{23}, -J_{13}, -J_{03}, -J_{12}, -J_{02}, -J_{01}; -z)\,. \tag{3.15}$$

D^q_0 transforms quantum algebras according to

$$D^q_0 : U_q\mathfrak{g}_{(\kappa_1,\kappa_2,\kappa_3)} \to U_q\mathfrak{g}_{(\kappa_3,\kappa_2,\kappa_1)}\,. \tag{3.16}$$

The 24 dualities existing in the classical case can be also generalized in a similar way. For certain values of the κ_i new deformations (related with different elections of the primitive generators) can be obtained by applying a duality to a known deformation. From this point of view, the multiplicity of geometries linked with a certain classical group somehow announces the existence of different quantizations; q-dualities play a similar role connecting deformed quantum algebras as their classical counterparts relate geometries [14].

4. (2+1) quantum kinematical algebras

The (2+1) kinematical groups of Bacry and Lévy–Leblond [29] appear in the 3D-CKG system. In terms of the physical generators (temporal translation H, spatial translations P_1, P_2, pure inertial transformations K_1, K_2 and spatial rotation J) these algebras can be identified to CK algebras following the three different assignations collected in table 1.

Table 1. Kinematical assignations.

Type	$(\kappa_1, \kappa_2, \kappa_3)$		H	P_1	P_2	K_1	K_2	J
(a)	$(\kappa_1, \kappa_2, +)$	$\kappa_2 \leqslant 0$	J_{01}	J_{02}	J_{03}	J_{12}	J_{13}	J_{23}
(b)	$(\kappa_1, +, \kappa_3)$	$\kappa_3 \leqslant 0$	J_{03}	J_{01}	J_{02}	J_{13}	J_{23}	J_{12}
(c)	$(\kappa_1, -, -)$		J_{02}	J_{03}	J_{01}	J_{23}	J_{12}	$-J_{13}$

Type (a) includes three 'relativistic' groups: Poincaré and two de Sitter; and their 'non-relativistic' limits: Galilei and two Newton–Hooke. Both types (b) and (c) contain once the above three 'relativistic' groups and (b) also two para-Poincaré and Carroll groups (see figure 3). Thus, we get from our pattern *three* q-structures for Poincaré and de Sitter

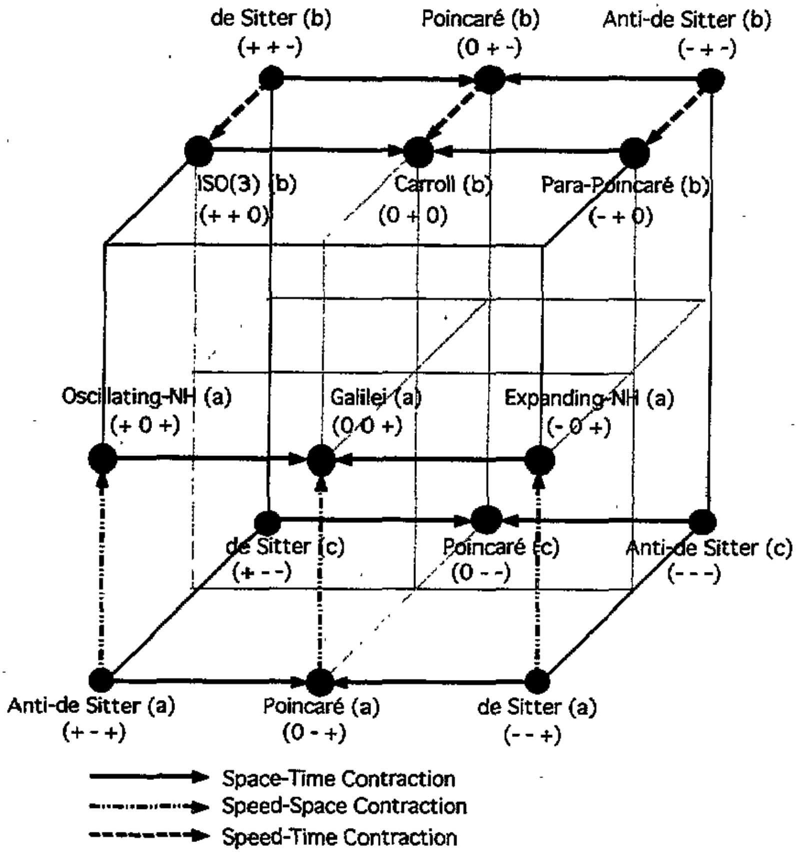

Figure 3. Each set of (2+1) kinematical groups is displayed in the CK diagram. Their physical standard names are used as well as an additional label (a, b, c) according to table 1.

algebras, but only *one* for Galilei and Newton–Hooke algebras. We consider each case in turn. (Hereafter we will consider $z \in \mathbb{R}$, in order to properly define its physical significance).

Type (a) assignation

(a1) q-deformed Poincaré algebra. The algebra with coefficients $(0, -, +)$ corresponds to the Poincaré algebra, with Minkowski space as $\mathcal{X}^{(0)}$, time-like lines as $\mathcal{X}^{(1)}$, and time-like planes as $\mathcal{X}^{(2)}$. The general expressions (3.4)–(3.8) when translated in terms of the 'physical generators' give the following Hopf structure:

Co-product:

$$\begin{aligned}
\Delta(P_2) &= 1 \otimes P_2 + P_2 \otimes 1 \\
\Delta(K_1) &= 1 \otimes K_1 + K_1 \otimes 1 \\
\Delta(H) &= \mathrm{e}^{-\frac{1}{2}zP_2} \otimes H + H \otimes \mathrm{e}^{\frac{1}{2}zP_2} \\
\Delta(P_1) &= \mathrm{e}^{-\frac{1}{2}zP_2} \otimes P_1 + P_1 \otimes \mathrm{e}^{\frac{1}{2}zP_2} \\
\Delta(K_2) &= \mathrm{e}^{-\frac{1}{2}zP_2} \otimes K_2 + K_2 \otimes \mathrm{e}^{\frac{1}{2}zP_2} + \mathrm{e}^{-\frac{1}{2}zP_2}(\tfrac{1}{2}zK_1) \otimes P_1 - P_1 \otimes \mathrm{e}^{\frac{1}{2}zP_2}(\tfrac{1}{2}zK_1) \\
\Delta(J) &= \mathrm{e}^{-\frac{1}{2}zP_2} \otimes J + J \otimes \mathrm{e}^{\frac{1}{2}zP_2} - \mathrm{e}^{-\frac{1}{2}zP_2}(\tfrac{1}{2}zK_1) \otimes H + H \otimes \mathrm{e}^{\frac{1}{2}zP_2}(\tfrac{1}{2}zK_1) .
\end{aligned} \tag{4.1}$$

The co-unit $\epsilon(X) = 0$ for all generators.
Antipode:

$$\gamma \begin{pmatrix} H \\ P_1 \\ P_2 \\ K_1 \\ K_2 \\ J \end{pmatrix} = - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -z & 0 & 0 & 0 & 1 & 0 \\ 0 & z & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} H \\ P_1 \\ P_2 \\ K_1 \\ K_2 \\ J \end{pmatrix} . \tag{4.2}$$

Non-zero commutation relations:

$$\begin{aligned} &[J, K_1] = K_2 \qquad [J, K_2] = -K_1 C_{-z^2}(P_2) \qquad [J, P_1] = S_{-z^2}(P_2) \\ &[J, P_2] = -P_1 \qquad [K_1, H] = P_1 \qquad [K_2, H] = S_{-z^2}(P_2) \\ &[K_1, P_1] = H \qquad [K_2, P_2] = H \qquad [K_1, K_2] = -J . \end{aligned} \tag{4.3}$$

Two central elements can be easily derived from (3.10):

$$C_1^q = 4S^2_{-z^2}(\tfrac{1}{2}P_2) + P_1^2 - H^2 \qquad C_2^q = S_{-z^2}(P_2)K_1 - HJ - P_1 K_2 . \tag{4.4}$$

The q-algebra automorphisms q-parity and q-time-reversal are defined by

$$\begin{aligned} &S^q_{(1)} \equiv \Pi^q : \{H \to H,\ P_i \to -P_i,\ J \to J,\ K_i \to -K_i;\ z \to -z\} \\ &S^q_{(0)} \cdot S^q_{(1)} \equiv \Theta^q : \{H \to -H,\ P_i \to P_i,\ J \to J,\ K_i \to -K_i;\ z \to z\} . \end{aligned} \tag{4.5}$$

The product zP_2 has to be dimensionless in order to have a homogeneous co-product. In this case, z has the dimension of length, in a way consistent with its behaviour under q-parity and q-time-reversal (4.5). It is also worth remarking that there exists a non-trivial Hopf subalgebra generated by $\langle P_2, P_1, H, K_1\rangle$ in which P_2 is a central element. If we think of assignation (a) as taking time-like planes as elements in $\mathcal{X}^{(2)}$, the presence of this subalgebra is rather natural since $\langle P_1, H, K_1\rangle$ is just the classical isotopy subalgebra of a time-like plane.

(a2) *q-deformed Galilei algebra.* For $(\kappa_1, \kappa_2, \kappa_3)$ equal to $(0, 0, +)$ we get a deformation of the (2+1) Galilei algebra. The co-product is also given by (4.1); the co-unit remains (3.5); the antipode turns into

$$\gamma \begin{pmatrix} H \\ P_1 \\ P_2 \\ K_1 \\ K_2 \\ J \end{pmatrix} = - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & z & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} H \\ P_1 \\ P_2 \\ K_1 \\ K_2 \\ J \end{pmatrix} \tag{4.6}$$

and commutation relations are

$$\begin{aligned} &[J, K_1] = K_2 \qquad [J, K_2] = -K_1 C_{-z^2}(P_2) \qquad [J, P_1] = S_{-z^2}(P_2) \\ &[J, P_2] = -P_1 \qquad [K_1, H] = P_1 \qquad [K_2, H] = S_{-z^2}(P_2) \\ &[K_1, P_1] = 0 \qquad [K_2, P_2] = 0 \qquad [K_1, K_2] = 0 . \end{aligned} \tag{4.7}$$

Three deformed brackets (like in (4.3)) are preserved. The q-Casimirs are

$$C_1^q = 4S^2_{-z^2}(\tfrac{1}{2}P_2) + P_1^2 \qquad C_2^q = S_{-z^2}(P_2)K_1 - P_1K_2\,. \tag{4.8}$$

In this case, a maximal Hopf subalgebra is $\langle H, P_1, P_2, K_1, K_2\rangle$. The quantum Galilei algebra can be obviously seen as a q-axial contraction (3.14) (q-speed-space contraction in physical terms) of the previous q-Poincaré algebra by making the following transformation:

$$(\mathbb{H}, \mathbb{P}_1, \mathbb{P}_2, \mathbb{K}_1, \mathbb{K}_2, \mathbb{J}; w) \equiv (H, \varepsilon P_1, \varepsilon P_2, \varepsilon K_1, \varepsilon K_2, J; z/\varepsilon) \tag{4.9}$$

and then taking the limit $\varepsilon \to 0$. Since $w\mathbb{P}_2 = zP_2$ and $w\mathbb{K}_1 = zK_1$, it is straightforward to check that the co-product is invariant under this contraction provided w is the new deformation parameter. The same is true for the co-unit. For the antipode we have

$$\begin{aligned} \gamma(\mathbb{K}_2) &= \varepsilon\gamma(K_2) = \varepsilon(zH - K_2) = \varepsilon(w\varepsilon\mathbb{H} - (1/\varepsilon)\mathbb{K}_2) \stackrel{\varepsilon\to 0}{\longrightarrow} -\mathbb{K}_2 \\ \gamma(\mathbb{J}) &= \gamma(J) = (-zP_1 - J) = (-w\varepsilon(1/\varepsilon)\mathbb{P}_1 - \mathbb{J}) \stackrel{\varepsilon\to 0}{\longrightarrow} -w\mathbb{P}_1 - \mathbb{J} \end{aligned} \tag{4.10}$$

which coincides with (4.6) in terms of the new deformation parameter w.

As far as the commutation relations are concerned, the non-deformed ones are easily computed and lead to the usual Galilei brackets after the limit $\varepsilon \to 0$. To obtain the deformed ones, we recall (2.7)

$$\begin{aligned} [\mathbb{K}_2, \mathbb{H}] &= \varepsilon S_{-z^2}(P_2) = \varepsilon S_{-\varepsilon^2 w^2}((1/\varepsilon)\mathbb{P}_2) = S_{-w^2}(\mathbb{P}_2) \stackrel{\varepsilon\to 0}{\longrightarrow} S_{-w^2}(\mathbb{P}_2) \\ [\mathbb{J}, \mathbb{K}_2] &= -\varepsilon K_1 C_{-z^2}(P_2) = -\varepsilon(1/\varepsilon)\mathbb{K}_1 C_{-\varepsilon^2 w^2}((1/\varepsilon)\mathbb{P}_2) \\ &= -\mathbb{K}_1 C_{-w^2}(\mathbb{P}_2) \stackrel{\varepsilon\to 0}{\longrightarrow} -\mathbb{K}_1 C_{-w^2}(\mathbb{P}_2) \end{aligned} \tag{4.11}$$

and the same procedure gives $[\mathbb{J}, \mathbb{P}_1] = S_{-w^2}(\mathbb{P}_2)$ due to the invariant behaviour of the generalized sine and cosine functions in (4.11). If we apply the transformation (4.9) to the Poincaré invariants (4.4), the contraction limit with the standard rescaling $1/\varepsilon^2$ gives the quantum Galilei Casimirs (4.8). So, there is a complete equivalence of this IW process and the limit $\kappa_2 \to 0$.

(a3) *Quantum de Sitter algebras.* The two de Sitter algebras correspond to $(\pm, -, +)$. The expressions for co-product, deformed commutations relations, etc, are readily obtained from (3.4)–(3.8), where the generalized sine $S_{-\kappa_1\kappa_3}(X)$ and cosine $C_{-\kappa_1\kappa_3}(X)$ functions are substituted by hyperbolic or circular functions (anti-de Sitter and de Sitter cases, respectively) in accordance with the sign of its label $-\kappa_1\kappa_3$.

(a4) *Quantum Newton–Hooke algebras.* Two Newton–Hooke (NH) q-algebras are obtained: $(+, 0, +)$ is associated to the oscillating-NH algebra, while $(-, 0, +)$ corresponds to the expanding-NH algebra. Since the co-product does not depend on κ_2, its expression will be the same for the three algebras belonging to a 'κ_2-column' in figure 1. This means that the q-axial contraction ($\kappa_2 \to 0$) leaves this co-product invariant; this was the case for the Poincaré and Galilei co-products studied in (a1) and (a2). The explicit computation of the de Sitter $\to$ Newton–Hooke contraction is similar to (4.11). The co-product of the q-oscillating-NH algebra coincides with the one defined for the q-anti-de Sitter $U_q(so(2, 2))$ algebra, and the same is true for the other pair (expanding-NH/de Sitter $U_q(so(3, 1))$) of kinematical algebras.

Type (b) q-Poincaré and q-de Sitter algebras

The CK algebra with $(0,+,-)$ corresponds to Minkowski spacetime ($\mathcal{X}^{(0)}$), with space-like lines as $\mathcal{X}^{(1)}$ and space-like planes as $\mathcal{X}^{(2)}$. The specialization of (3.4)–(3.8) to $(\kappa_i) \equiv (0,+,-)$ and assignation (b) give a q-deformed Poincaré algebra that is *essentially* different from the one studied in (a). The kinematical automorphisms Π^q, Θ^q are equal, in this order, to the involutions $S^q_{(0)} \cdot S^q_{(2)}$ and $S^q_{(2)}$; z has now the character of a time. This process leads to the following Hopf structure:

$$
\begin{aligned}
&\Delta(H) = 1 \otimes H + H \otimes 1 \qquad \Delta(J) = 1 \otimes J + J \otimes 1 \\
&\Delta(P_1) = e^{-\frac{1}{2}zH} \otimes P_1 + P_1 \otimes e^{\frac{1}{2}zH} \qquad \Delta(P_2) = e^{-\frac{1}{2}zH} \otimes P_2 + P_2 \otimes e^{\frac{1}{2}zH} \\
&\Delta(K_1) = e^{-\frac{1}{2}zH} \otimes K_1 + K_1 \otimes e^{\frac{1}{2}zH} - e^{-\frac{1}{2}zH}(\tfrac{1}{2}zJ) \otimes P_2 + P_2 \otimes e^{\frac{1}{2}zH}(\tfrac{1}{2}zJ) \\
&\Delta(K_2) = e^{-\frac{1}{2}zH} \otimes K_2 + K_2 \otimes e^{\frac{1}{2}zH} + e^{-\frac{1}{2}zH}(\tfrac{1}{2}zJ) \otimes P_1 - P_1 \otimes e^{\frac{1}{2}zH}(\tfrac{1}{2}zJ)
\end{aligned}
\tag{4.12}
$$

$$
\gamma \begin{pmatrix} H \\ P_1 \\ P_2 \\ K_1 \\ K_2 \\ J \end{pmatrix} = - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -z & 0 & 1 & 0 & 0 \\ 0 & 0 & -z & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} H \\ P_1 \\ P_2 \\ K_1 \\ K_2 \\ J \end{pmatrix} . \tag{4.13}
$$

The non-vanishing commutation relations are

$$
\begin{aligned}
&[J, K_1] = K_2 \qquad [J, K_2] = -K_1 \qquad [J, P_1] = P_2 \qquad [J, P_2] = -P_1 \\
&[K_1, H] = P_1 \qquad [K_2, H] = P_2 \qquad [K_1, P_1] = S_{-z^2}(H) \\
&[K_2, P_2] = S_{-z^2}(H) \qquad [K_1, K_2] = -JC_{-z^2}(H) .
\end{aligned}
\tag{4.14}
$$

This is just the (2+1)-dimensional version described in [30] of the q-Poincaré algebra of Lukierski *et al* [16]. It can be checked that $\langle H, J, P_1, P_2 \rangle$ is a Hopf subalgebra where H is now a central element. This deformed subalgebra corresponds to a quantization of a direct sum between H and the classical isotopy subalgebra $\langle J, P_1, P_2 \rangle$ of the space-like planes chosen in this assignation as $\mathcal{X}^{(2)}$.

The Casimir elements of this algebra are

$$
C_1^q = 4S^2_{-z^2}(\tfrac{1}{2}H) - P_1^2 - P_2^2 \qquad C_2^q = S_{-z^2}(H)J - P_2K_1 + P_1K_2 . \tag{4.15}
$$

The element C_1^q has been used to study the strong interaction in nuclei [25]; in that model, the deformation parameter is linked with a fundamental time scale.

The de Sitter quantum kinematical algebras in type (b) correspond to $(\pm,+,-)$ and can be obtained by the same procedure; similar comments concerning the dimensional properties of z can be done.

Type (c) q-Poincaré The CK algebra with the triad $(0,-,-)$ is the Poincaré algebra realized in Minkowski spacetime ($\mathcal{X}^{(0)}$) with space-like lines ($\mathcal{X}^{(1)}$) and time-like planes ($\mathcal{X}^{(2)}$). For $(\kappa_i) \equiv (0,-,-)$ and assignation (c), relations (3.4)–(3.8) gives a q-deformed Poincaré algebra. The kinematical automorphisms Π^q, Θ^q are equal to the involutions $S^q_{(0)} \cdot S^q_{(1)} \cdot S^q_{(2)}$

and $S^q_{(1)} \cdot S^q_{(2)}$, respectively. Hence, z has again the character of a length. The Hopf structure is

$$
\begin{aligned}
&\Delta(P_1) = 1 \otimes P_1 + P_1 \otimes 1 \qquad \Delta(K_2) = 1 \otimes K_2 + K_2 \otimes 1 \\
&\Delta(H) = e^{-\frac{1}{2}zP_1} \otimes H + H \otimes e^{\frac{1}{2}zP_1} \qquad \Delta(P_2) = e^{-\frac{1}{2}zP_1} \otimes P_2 + P_2 \otimes e^{\frac{1}{2}zP_1} \\
&\Delta(K_1) = e^{-\frac{1}{2}zP_1} \otimes K_1 + K_1 \otimes e^{\frac{1}{2}zP_1} + e^{-\frac{1}{2}zP_1}(\tfrac{1}{2}zK_2) \otimes P_2 - P_2 \otimes e^{\frac{1}{2}zP_1}(\tfrac{1}{2}zK_2) \\
&\Delta(J) = e^{-\frac{1}{2}zP_1} \otimes J + J \otimes e^{\frac{1}{2}zP_1} + e^{-\frac{1}{2}zP_1}(\tfrac{1}{2}zK_2) \otimes H - H \otimes e^{\frac{1}{2}zP_1}(\tfrac{1}{2}zK_2)
\end{aligned}
\tag{4.16}
$$

$$
\gamma \begin{pmatrix} H \\ P_1 \\ P_2 \\ K_1 \\ K_2 \\ J \end{pmatrix} = - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -z & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -z & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} H \\ P_1 \\ P_2 \\ K_1 \\ K_2 \\ J \end{pmatrix} . \tag{4.17}
$$

The non-vanishing commutation relations are:

$$
\begin{aligned}
&[J, K_1] = K_2 C_{-z^2}(P_1) \qquad [J, K_2] = -K_1 \qquad [J, P_1] = P_2 \\
&[J, P_2] = -S_{-z^2}(P_1) \qquad [K_1, H] = S_{-z^2}(P_1) \qquad [K_2, H] = P_2 \\
&[K_1, P_1] = H \qquad [K_2, P_2] = H \qquad [K_1, K_2] = -J
\end{aligned}
\tag{4.18}
$$

and the q-Casimirs are

$$
C_1^q = 4S^2_{-z^2}(\tfrac{1}{2}P_1) + P_2^2 - H^2 \qquad C_2^q = S_{-z^2}(P_1)K_2 + HJ - P_2K_1 . \tag{4.19}
$$

This quantum algebra includes again a Hopf subalgebra containing the isotopy subalgebra of a 'time-like plane' $\langle K_2, P_2, H\rangle$ and P_1; the latter being a central element. Its properties are essentially the same as for the type (a) Poincaré deformed algebra.

5. Concluding remarks

It is remarkable that three different deformations for Poincaré algebra are obtained at once within our scheme. Type (a) and (c) algebras have primitive generators with a similar physical relevance: P_2, K_1 and P_1, K_2 respectively. However, type (b) has H, J as primitive elements. In some sense (a)/(c) and (b) can be considered as kinds of 'space-like' and 'time-like' q-deformations.

In contrast, only one q-Galilei algebra, corresponding to a q-speed-space contraction of the type (a) q-Poincaré algebra, arises in our approach. Another quantum Galilei algebra with H and J as primitive generators can be obtained by contracting the type (b) q-Poincaré in the same way as Giller *et al* [30] do with the (3+1) q-Poincaré algebra. Nevertheless, the resulting q-Galilei has no deformed commutation relations, in spite of the preservation of the co-product (4.12). Note that this quantum contraction is not a CK one (compare with (3.14)); this explains the absence of this q-Galilei algebra in our scheme. The CK framework implies that the natural non-relativistic limit of type (b) q-Poincaré algebra is a q-Carroll algebra. This process can be carried out by means of a q-speed-time contraction (see figure 3).

The three-dimensional Euclidean algebra $E(3)$ appears in a natural way in the CK structure as the $iso(3)$ algebra with κ_i parameters $(0, +, +)$. Thus, a quantum Euclidean algebra will be obtained from (3.4)–(3.8) for these values of κ_i by identifying the J_{0i} as the translation generators P_i and keeping the remaining J_{ij} as the rotation ones with the standard labelling. It is easy to check that the final expressions are just the same as the ones given by Celeghini *et al* in [17].

Finally, we would like to mention some open problems in this context. First, it would be worthwhile to analyse the possible deformations of the non-trivial centrally extended kinematical algebras (Galilei, Newton–Hooke and Carroll algebras) as has been done for the (1+1) case [14]. In particular, the extended oscillating Newton–Hooke algebra would be a proper two-dimensional quantum oscillator. Another interesting problem would consist of extending the CK scheme of deformation to higher dimensions. In spite of the fact that the co-product for the (2+1)-dimensional case (now dependent on κ_i) is rather different from the (1+1) co-product, the (1+1) and (2+1) cases show some deep common properties: both structures are autodual, the whole geometrical underlying structure is generalized in the same way, and the antipode is essentially the same for both cases. It is therefore not unreasonable to assume that these common properties should provide a way to understand how the CK scheme of deformation extends to higher dimensions. Work on these lines is in progress.

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